WAVES IN SPATIALLY-DISORDERED NEURAL FIELDS: A CASE STUDY IN UNCERTAINTY QUANTIFICATION

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Abstract. Neural field models have been used for many years to model a variety of macroscopic spatiotemporal patterns in the cortex. Most authors have considered homogeneous domains, resulting in equations that are translationally invariant. However, there is an obvious need to better understand the dynamics of such neural field models on heterogeneous domains. One way to include heterogeneity is through the introduction of randomly-chosen "frozen" spatial noise to the system. In this chapter we investigate the effects of including such noise on the speed of a moving "bump" of activity in a particular neural field model. The spatial noise is parameterised by a large but finite number of random variables, and the effects of including it can be determined in a computationally-efficient way using ideas from the field of Uncertainty Quantification. To determine the average speed of a bump in this type of heterogeneous domain involves evaluating a high-dimensional integral, and a variety of methods are compared for doing this. We find that including heterogeneity of this form in a variety of ways always slows down the moving bump.

1. Introduction

Neural field models have been used for many years as models of large-scale pattern formation in the cortex [9, 13, 1, 32, 33, 35, 16, 28]. These models are typically formulated as nonlocal partial differential equations in space and time where the nonlocality arises via spatial integrals, meant to represent the influence of neurons at many different spatial locations on the dynamics at a specific location [13, 9]. They have been used to model a variety of neurophysiological phenomena such as working memory [33], orientation tuning in the visual cortex [4] and EEG rhythms [39]. Much of the analysis of patterns in these models has assumed that the domain is homogeneous and thus the heterogeneity such as adding “frozen” spatial noise, and driving the system with temporal noise. Bressloff [8] adapted ideas from PDE theory to study the effects of slowly modulated (in space) synaptic connectivity on the invasion and extinction of activity in a neural field model. Several authors have very recently considered the effects of additive spatio-temporal noise on the dynamics of a neural field [11, 26, 24, 6].

In this chapter we will use ideas from the relatively new field of Uncertainty Quantification (UQ) to investigate the effects of spatial heterogeneity on the dynamics of moving “bumps” in a particular neural field model. Traditionally, numerical models of physical phenomena have been solved under the assumption that both the initial conditions and all values of relevant parameters are known exactly. However, recent increases in computational power have meant that it is now possible to solve a model where one or more parameters are not known exactly, but are known (or assumed) to come from some distribution(s). For our purposes, UQ involves a systematic investigation of the effects of this uncertainty in parameter values on quantities of interest. The field of UQ is large and rapidly growing [34, 45, 42] and here we will only use those aspects of it which are directly relevant.

2. Model and Analysis

The model we first consider is governed by the following equations:

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_0^{2\pi} G(x-y)F[u(y,t)-a(y,t)+h(y)]dy \tag{1}$$

$$\tau\frac{\partial a(x,t)}{\partial t} = Bu(x,t) - a(x,t) \tag{2}$$

where $u(x,t)$ represents the average voltage of neurons at position $x \in [0,2\pi]$ at time t, and $a(x,t)$ represents the value of a slow variable at x and t which provides negative random function of y, in a way to be explained below. In particular we wish to answer the question: given that h(y) is randomly chosen from some distribution of functions, what is the expected value of the average speed of the resulting travelling bump (after transients have decayed)? As mentioned, we will answer this using techniques from the field of uncertainty quantification [34, 41]. Here, the uncertainty arises because we do not exactly know h(y). This uncertainty then affects the dynamics of the neural field model, making measurable quantities such as the bump speed uncertain, i.e. have some distribution of values. Typically, we would like to describe this distribution so that we can calculate, for example, its mean.

The form of the coupling function G(x) allows us to write (1) as

(4)
$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= -u(x,t) + 0.09\int_0^{2\pi} F[u(y,t) - a(y,t) + h(y)]\,dy \\ &+ 0.45\cos x\int_0^{2\pi} F[u(y,t) - a(y,t) + h(y)]\cos y\,dy \\ &+ 0.45\sin x\int_0^{2\pi} F[u(y,t) - a(y,t) + h(y)]\sin y\,dy\end{aligned}$$

As noted [31], if we expand u(x, t) and a(x, t) in Fourier series in x we see that terms of the form sin (nx) and cos (nx) for n > 1 will decay to zero, and since we are not interested in transients we write

(5)
$$u(x,t) = u^0(t) + u^c(t)\cos x + u^s(t)\sin x$$

and

(6)
$$a(x,t) = a^0(t) + a^c(t)\cos x + a^s(t)\sin x$$

Substituting these expansions into (2) and (4) we find that the modulated bumps of interest are described by the six ordinary differential equations (ODEs)

(7)
$$\frac{du^0}{dt} = -u^0 + 0.09\int_0^{2\pi} F[u^0 - a^0 + (u^c - a^c)\cos x + (u^s - a^s)\sin x + h(x)]\,dx$$

(8)
$$\frac{du^c}{dt} = -u^c + 0.45\int_0^{2\pi} F[u^0 - a^0 + (u^c - a^c)\cos x + (u^s - a^s)\sin x + h(x)]\cos x\,dx$$

$$\frac{du^s}{dt} = -u^s + 0.45\int_0^{2\pi} F[u^0 - \ldots x + (\ldots$$ [illegible]

$$0 = \mathbf{B}u^0(t_j) - a^0(t_j) + \tau \ \mathbf{M}$$

To find the eigenpairs of C consider the function $\cos(my)$, where $m \in \mathbb{N}^+$. This is periodic on the domain $[0, 2\pi]$ and we have

$$\text{(29)} \qquad 2b \int_0^{2\pi} C(x,y)\cos(my)\,dy = \sigma \int_0^{2\pi} \exp\left[-\frac{\pi}{4}\left(\frac{x-y}{b}\right)^2\right] \cos(my)\,dy$$

$$\text{(30)} \qquad = \sigma \int_{x-2\pi}^{x} \exp\left[-\frac{\pi}{4}\left(\frac{z}{b}\right)^2\right] \cos(m(x-z))\,dz$$

Now if b is small relative to the domain size (2π), we can approximate this integral by the infinite one:

$$\text{(31)} \qquad 2b \int_0^{2\pi} C(x,y)\cos(my)\,dy \approx \sigma \int_{-\infty}^{\infty} \exp\left[-\frac{\pi}{4}\left(\frac{z}{b}\right)^2\right] \cos(m(x-z))\,dz$$

$$= \sigma\cos(mx) \int_{-\infty}^{\infty} \exp\left[-\frac{\pi}{4}\left(\frac{z}{b}\right)^2\right] \cos(mz)\,dz$$

$$\text{(32)} \qquad + \sigma\sin(mx) \int_{-\infty}^{\infty} \exp\left[-\frac{\pi}{4}\left(\frac{z}{b}\right)^2\right] \sin(mz)\,dz$$

$$\text{(33)} \qquad = 2b\sigma\cos(mx)\exp\left[\frac{-(mb)^2}{\pi}\right]$$

where we have used the fact that [38]

$$\text{(34)} \qquad \int_{-\infty}^{\infty} \exp\left[-\frac{\pi}{4}\left(\frac{z}{b}\right)^2\right] \cos(mz)\,dz = 2b\exp\left[\frac{-(mb)^2}{\pi}\right]$$

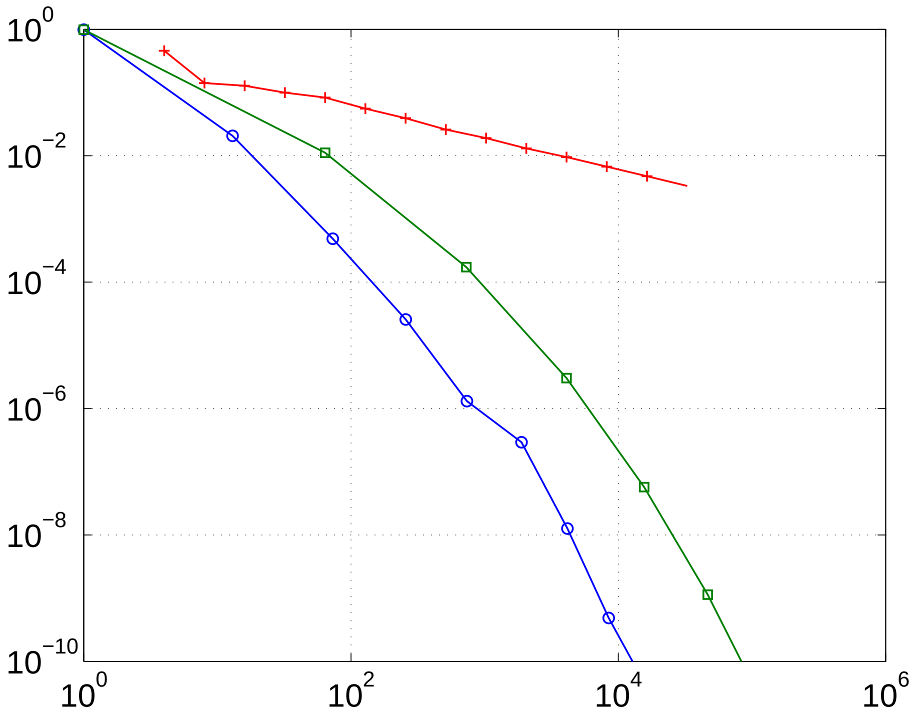
and that $\exp[-(\pi/4)(z/b)^2]\sin(mz)$ is an odd function. Thus (keeping in mind the approximations made above) a partial set of eigenvalues and eigenfunctions for C is

$$\text{(35)} \qquad \lambda_m^{(1)} = \sigma\exp\left[\frac{-(mb)^2}{\pi}\right]; \qquad e_m^{(1)}(x) = \frac{\cos(mx)}{\sqrt{\pi}}$$

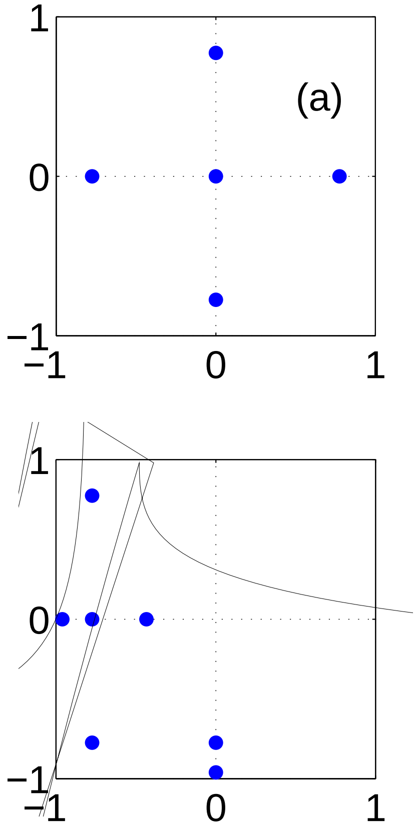
for $m = 1, 2$

form of the polynomials is determined by the probability density function of the random variables, the $\{\beta_m\}$. [44, 42]) Once the coefficients have been found, any quantity such as the expected value of, say u_0^0, can be found by integrating over the space of random variables. Unfortunately, modifying code capable of solving (19)-(24) and (25) to find all coefficients in the expansion just mentioned is non-trivial.

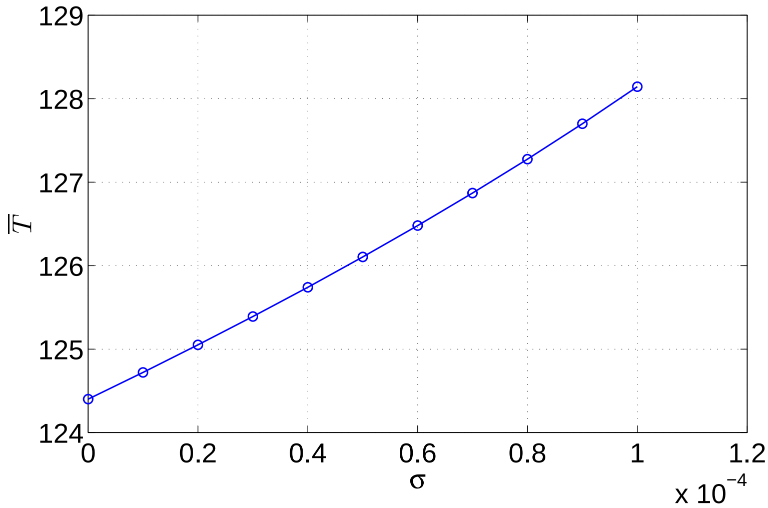
The other common alternative is referred to as stochastic collocation [41, 34], which involves solving (19)-(24) and (25) at a number of different points in the random parameter space, i.e. using different $\{\beta_m\}$. We then have the value of all variables $u_0^0, u_1^0, \ldots, a_{2M}^s, T$ at these different points and can use interpolation to estimate the values of these variables at other points in the random parameter space. If the values of $\{\beta_m\}$ at which (19)-(24) and (25) are solved are chosen appropriately, the solutions of these equations at these

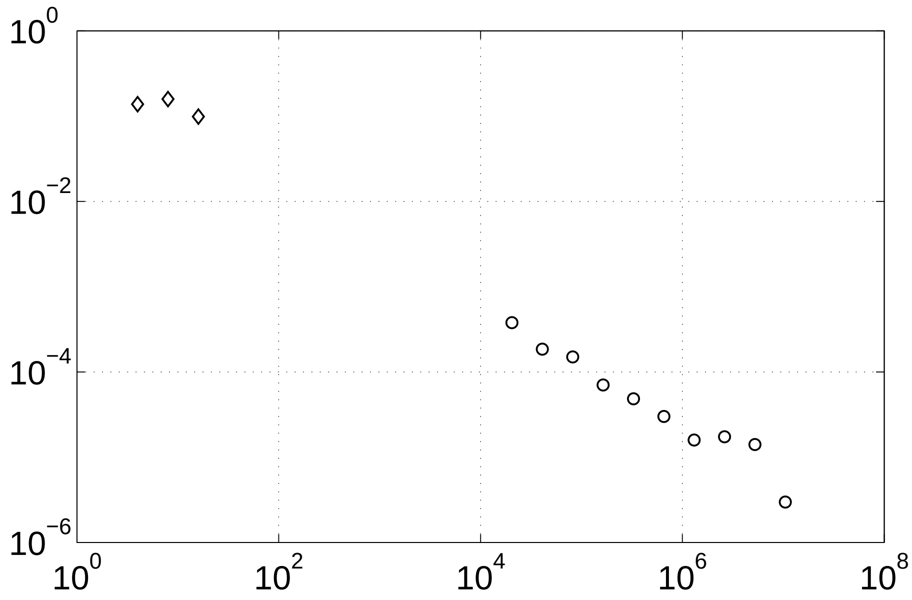


10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{0}
10^{2}
10^{4}
10^{6}



(a)
(b)
1
0
−1
−1
0
1


129
128
127
126
125
124
$\overline{T}$
0
0.2
0.4
0.6
0.8
1
1.2
σ
x 10^{-4}



10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{0}
10^{2}
10^{4}
10^{6}
10^{8}

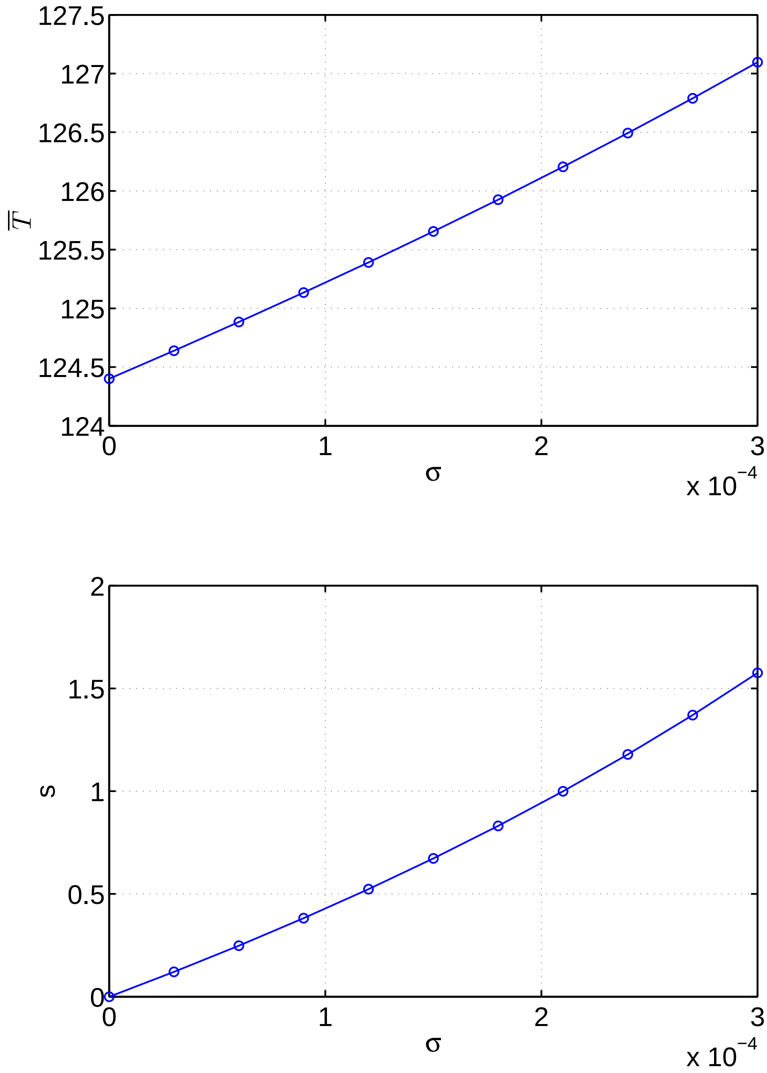


Figure 11. **Modulated connectivity. Mean period, $\overline{T}$ (top) and standard deviation, $s = \sqrt{V_T}$ (bottom) as a function of random field strength σ.**

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