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perturbation was added at $t = 10$ and we see that the solution quickly settled to being even about an x value of approximately $0.87 \times 2\pi$. Note that during the transients the first modes of the real and imaginary parts of A are even about different points, but on the attractor they are even about the same point, as they must be for an even solution. See [19] for more details on orbital stability.

Despite searching, we could not find any evidence of a "blowout" bifurcation [4] in which the even solution remains chaotic while the dominant Lyapunov exponent in the normal direction changes from zero to positive as a parameter is varied. The reason for this is that, as shown in Figure 7, the solution in the even subspace becomes periodic or quasiperiodic before the normal Lyapunov exponent becomes positive.

7 Conclusions

Our numerical results show that for much of the parameter space for the CGL equation, chaotic solutions which have some sort of reflectional symmetry are unstable to perturbations which break that symmetry, while there are also small regions of parameter space in which there are chaotic even solutions that are asymptotically stable with respect to odd perturbations. Of course we have not investigated all of the three-dimensional parameter space and there may be more interesting behaviour waiting to be found. Periodic solutions with symmetry are sometimes stable with respect to symmetry-breaking perturbations but it would appear that for most parameter values for the CGL equation that for arbitrary initial conditions, if the final solution is chaotic then it will have the minimum possible amount of symmetry. Clearly these ideas apply to any PDEs with symmetry and are not restricted to the CGL equation. Different results may be obtained for different equations.

Acknowledgements

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which are, for Matlab, typically smaller than the quantity being transformed by a factor of 10^{16}. Thus, for example, the discrete Fourier transform of a real, even function will not be purely real, but will have a small imaginary component, and when the inverse transform is taken the result will have a small odd component. If this even solution is stable with respect to odd perturbations this will not matter, but for solutions such as those shown in Figures 1–2 where we have instability to odd perturbations these errors will grow exponentially in time and ultimately overwhelm the even solution.

If we want the solution to remain within an invariant subspace we must modify the numerical scheme. For the above example, this is simply done by setting the imaginary component of the transformed variable to zero immediately after it is calculated, or better still, only working with the real part of the transformed variable. This is the technique used for Lyapunov exponent calculations. This method cannot, however, be used in, for example, the calculations for Figure 1. Here, although we added a small odd perturbation after 2 seconds to demonstrate instability in this isotypic component, strictly speaking this was not necessary as, given sufficient time, the numerical errors introduced by the Fourier transform would have grown large enough to destroy the appearance of evenness — all we did was hasten the onset of this phenomenon.

6.1 Stable even chaos

Although for the vast majority of parameter values we examined, solutions that were chaotic when restricted to lie in a fixed point subspace were unstable with respect to perturbations normal to that space, we did find a region in parameter space in which there exist chaotic solutions in the subspace of even solutions that are stable with respect to perturbations in the odd subspace. We plot the Lyapunov exponents in part of this region in Figure 7 and show a typical example, at parameter values $R = 1.05$, $\nu = 4$, $\mu = -4$, in Figure 8, where this particular solution is even about the origin. We might describe this as "weak" chaos arising from the bifurcation of a periodic or quasiperiodic orbit for which the dominant Lyapunov exponent in the normal direction remains zero as a parameter is varied.

The chaotic even solution at these parameter values is *orbitally stable*, meaning that there is a continuous family of such solutions related to one another by the spatial shift, r_α, and (assuming there are no coexisting attractors) an arbitrary initial condition will be attracted to one member of this family i.e. a solution that is even about some point. We demonstrate this in Figure 9 where we plot a solution started at a randomly chosen initial condition together with one of the two points in $[0, 2\pi)$ about which the first mode of the solution (i.e. that described by a linear combination of yp0 haot6 eao ey weak vti t 6 ex

(This is just the inverse discrete Fourier transform of $\{X_k(t)\}$.) The spectral coefficients, $X_k(t)$, are obtained from $\{A_r(x_n,t)\}$ via the discrete Fourier transform

$$X_k(t) = \sum_{n=1}^{N} A_r(x_n,t)\exp(-ikx_n), \qquad 0 \le k \le N-1. \tag{5.2}$$

Since $A_r(x_n,t)$ is real,

$$X_k(t) = \overline{X_{N-k}(t)}, \qquad k = 1,\dots,N/2-1,$$

and both $X_0(t)$ and $X_{N/2}(t)$ are real. Furthermore, if $A_r(x_n,t)$ is even then $\{X_k(t)\}$ is real, and if $A(x_n,t)$ is odd then $\{X_k(t)\}$ is purely imaginary.

We use $\{X_k(t)\}$ as our dependent variables, i.e. given $\{X_k(t)\}$ we want to know $\{\dot{X}_k(t)\}$. Using the spectral representation of our solution, (5.2), evaluation of the linear terms $RA + (1+i\nu)\nabla^2 A$ in (1.1) is simple, and unlike a finite-difference scheme, the spatial differentiation is exact. To calculate the nonlinear term $(1+i\mu)A|A|^2$ the inverse transform (5.1) is used to form $\{A(x_n,t)\}$ from its real and imaginary parts. The cubic term $\{A(x_n,t)|A(x_n,t)|^2\}$ is then calculated, and a Fourier transform of the form (5.2) is used to evaluate the contribution of this nonlinear term to $\{\dot{X}_k(t)\}$. Anti–aliasing is performed in the calculation of the nonlinear term using padding and truncation as described in [10] and N is chosen to be a power of 2 so that the fast Fourier transform and its inverse can be used.

6 Numerical Results

In Figure 1 we show an example of a periodic orbit that is stable within X_e but which is unstable to odd perturbations. We start from a randomly chosen even initial condition which evolves rapidly to a periodic state. A small odd perturbation is introduced at $t = 2$ and this grows exponentially in time so that for $t >$

dominant Lyapunov exponent associated with the attractor in X_e and in X but did not look at the effect of symmetry-breaking perturbations.

The group Σ_3 has four one-dimensional irreducible representations and so there are four corresponding isotypic components. These can be specified as

$$\begin{aligned}
W_1 &= \{A \in X : A(0,t) = 0,\ A_x(\pi/2,t) = 0\} \ = \ \mathrm{Fix}(\Sigma_3) \\
W_2 &= \{A \in X : A_x(0,t) = 0,\ A(\pi/2,t) = 0\} \\
W_3 &= \{A \in X : A(0,t) = 0,\ A(\pi/2,t) = 0\} \\
W_4 &= \{A \in X : A_x(0,t) = 0,\ A_x(\pi/2,t) = 0\},
\end{aligned}$$

orbit at the origin, we have that

$$L_\theta A(x,t) = \left.\frac{d(\theta A(x,t))}{d\theta}\right|_{\theta=0} = \left.\frac{d(e^{i\theta}A(x,t))}{d\theta}\right|_{\theta=0} = iA(x,t).$$

Similarly,

$$L_\alpha A(x,t) = A_x(x,t), \qquad L_\beta A(x,t) = A_t(x,t).$$

Thus, any chaotic attractor which is not spatially uniform has three zero Lyapunov exponents associated with it. If the attractor lies in $\mathrm{Fix}(\Sigma)$ for some subgroup Σ of Γ, then the zero Lyapunov exponents may occur in different Σ-isotypic components. These can be determined by finding which isotypic components contain the trajectories $LA(x,t)$ for each $L \in \mathcal{L}$. However, for any symmetry, $L_\theta A(x,t)$ and $L_\beta A(x,t)$ will always have the same symmetry as the solution trajectory and so occur in $W_1 = \mathrm{Fix}(\Sigma)$ whereas in some cases, $L_\alpha A(x,t)$ may occur in a different isotypic component.

As a simple example, consider the CGL equation with homogeneous Neumann boundary conditions which is equivalent to considering solutions which are invariant under the reflectional symmetry s_1. Thus, the symmetries of the solution are given by $\Sigma = \{I, s_1\} \simeq \mathbf{Z}_2$. The Σ-isotypic components are $W_1 = \mathrm{Fix}(\Sigma)$ which consists of all even periodic functions and W_2 which consists of all odd periodic functions. When only a reflectional symmetry is involved, the isotypic components W_1 and W_2 are often referred to as the symmetric and antisymmetric spaces respectively. In this case we see that $L_\theta A(x,t)$ and $L_\beta A(x,t)$ are symmetric functions while $L_\alpha A($

exponent associated with the isotypic component W_k can then be found using the *vector* form of the variational equation restricted to W_k given by

$$\dot{\phi}_k = D_k f(x(t))\phi_k, \qquad \phi_k(0) = w_k \in W_k, \tag{3.4}$$

and is given by

$$\lambda_{1,k} = \lim_{t\to\infty} \frac{1}{t} \log \|\phi_k(t)$$

3.1 Classification of Lyapunov exponents

For a general n–dimensional ODE

$$\dot{x} = f(x), \qquad x(0) = x_0, \tag{3.1}$$

we find the Lyapunov exponents by integrating the variational equation

$$\dot{\Phi} = Df(x(t))\Phi, \qquad \Phi(0) = I, \tag{3.2}$$

where $Df(x(t))$ is the Jacobian of f evaluated at $x(t)$, the solution of (3.1). We define

$$\Lambda = \lim_{t\to\infty} [\Phi(t)^T \Phi(t)]^{1/2t}, \tag{3.3}$$

provided this limit exists, where the superscript "T" denotes the matrix transpose. The Multiplicative Ergodic Theorem of Oseledec [18, 27] states that this limit exists for μ-almost all x_0, where μ is the invariant measure associated with the attractor of (3.1). If the eigenvalues of Λ are m_i, $i = 1, 2, \ldots, n$, then the Lyapunov exponents of the solution $x(t)$ are

$$\lambda_i = \log |m_i|, \quad i = 1, 2, \ldots, n.$$

If (3.1) is equivariant with respect to some compact Lie group Γ, then we have the following result [6].

Lemma 3.1 *Let S be an invariant set contained in* $\mathrm{Fix}(\Sigma)$ *for some subgroup Σ of Γ. For $x_0 \in S$, let $x(t)$ be the solution of (3.1) with $x(0) = x_0$. Then the solution $\Phi(t)$ of the variational equation (3.2) commutes with the action of Σ, i.e.*

$$\sigma\Phi(t) = \Phi(t)\sigma$$

for all $\sigma \in \Sigma$ and $t \geq 0$.

A corollary of this is that the matrix Λ defined by (3.3) also commutes with the action of Σ. Thus, Λ can be put into block diagonal form and so its eigenvalues, and thus the Lyapunov exponents, can be associated with particular isotypic components.

The dominant (most positive) Lyapunov exponent associated with each isotypic component is the most important one for our purposes since it indicates whether the invariant set S is stable with respect to perturbations associated with the particular isotypic component (a positive dominant Lyapunov exponent implies that S is unstable to such perturbations). Using the block diagonal form of Λ these are easily computed.

For a particular Σ-isotypic component W_k we know that $Df(x(t))$ leaves W_k invariant and so we denote its restriction to W_k by $D_k f(x(t)) : W_k \to W_k$. The dominant Lyapunov

The significance of this isotypic decomposition is that for a linear operator L satisfying (2.3), all the isotypic components are invariant under L, that is

$$L : W_k \to W_k.$$

This results in a block diagonal structure to the linear operator L.

We assume that W_1 is associated with the trivial irreducible representation $\gamma = I$ for all $\gamma \in \Gamma$ and so $W_1 = \mathrm{Fix}(\Gamma)$.

This is relevant to the calculation of Lyapunov exponents since the variational equation involves the linear operator $g_A(A)$. It is well known and easily verified that if there is a trajectory $A(t)$ of (2.1) such that $A(t) \in \mathrm{Fix}(\Sigma)$ for some subgroup Σ of Γ, then

$$\sigma g_A(A(t)) = g_A(A(t))\sigma \quad \text{for all } \sigma \in \Sigma.$$

Thus, the linear operator g_A decomposes on the Σ-isotypic components of the space X. Thus, the important symmetry group in this case is not the symmetry group Γ of the equation but the subgroup Σ of symmetries of the particular solution being considered.

We now consider the CGL equation (1.1) on the one–dimensional domain $[0, 2\pi)$ with periodic boundary conditions. This equation has both continuous and discrete symmetries which are given by

$$\begin{aligned}
\theta A(x,t) &= e^{i\theta} A(x,t), & \theta \in [0, 2\pi) \\
r_\alpha A(x,t) &= A(x+\alpha, t), & \alpha \in [0, 2\pi) \\
\tau_\beta A(x,t) &= A(x, t+\beta), & \beta \in \mathbb{R} \\
s_1 A(x,t) &= A(-x, t). &
\end{aligned}$$

These symmetries correspond respectively to a rotation of the complex amplitude, space translation, time translation and a spatial reflection. We note that a special case of the rotation occurs when $\theta = \pi$ and this gives another symmetry of order two. Since this will be important in our later work, we define

$$\pi A(x,t) := s_2 A(x,t) = -A(x,t).$$

Relative equilibria are associated with continuous symmetries and in this case, the θ symmetry gives rise to such solutions which are often referred to as rotating waves. These were studied in some detail in [15] which included a linear stability analysis.

3 Lyapunov Exponents and Symmetry

The way that symmetry affects the determination of Lyapunov exponents was considered in [6] and applied to systems of coupled oscillators. We briefly review the main results of that work and will then apply the ideas to the CGL equation.

2 Symmetries of the CGL Equation

In this section we briefly outline some of the theory of dynamical systems with symmetry, concentrating on its applicability to the CGL equation. Group theory is the natural language with which to discuss symmetry; see [19] for many results concerning the application of symmetry to dynamical systems and their bifurcations.

We consider a general evolution equation of the form

$$A_t = g(A), \qquad g : X$$

if this motion is chaotic, then we can determine normal Lyapunov exponents associated

1 Introduction

Pattern formation in nonlinear partial differential equations is a much studied topic. One common problem is determining the spatial patterns which can occur when a spatially uniform (steady) state loses stability. In 1983, Yamada and Fujisaka [33] were interested in the stability of spatially uniform *chaotic* solutions of a nonlinear partial differential equation to perturbations which are not spatially uniform. In order to study this problem, they considered a finite difference discretisation of the PDE which gave a finite-dimensional system of coupled oscillators. The uniform state for the PDE corresponds to a synchronised state for the coupled oscillators. Stability of this uniform state was described in terms of what we now call normal Lyapunov exponents. However, the only numerical results presented in this work were for two coupled Lorenz systems.

This work went largely unnoticed until Pecora and Carroll [29] demonstrated that in some circumstances it is possible to synchronise two identical chaotic systems by linking them with a common signal. Since that time, there has been much interest in and study of synchronisation in systems of coupled oscillators, one interesting application being secure communication [26].

Mathematically speaking, synchronisation corresponds to motion in an invariant subspace which is stable with respect to perturbations normal to the subspace. If the largest normal Lyapunov exponent, associated with perturbations normal to the subspace, is negative, then the synchronised state has a positive measure basin of attraction, which may however be riddled so that there is a dense set of positive measure in any neighbourhood of the invariant subspace which is in the basin of another attractor [1, 2, 3, 28]. If the largest normal Lyapunov exponent changes sign as a parameter is varied, then a blowout bifurcation occurs which may be either supercritical or subcritical [2]. In order to gain a deeper understanding of these phenomena, model equations of low dimension are often studied [4].

The synchronised state of coupled identical oscillators is a natural setting for an invariant subspace. Another natural way of generating invariant subspaces is by the use of symmetry. Fixed point spaces are invariant under the dynamics of a system with symmetry and blowout bifurcations can be considered from these invariant subspaces also. Rings of coupled oscillators with symmetry were considered by Aston and Dellnitz [6] and it was shown how the normal Lyapunov exponents can be classified according to the symmetry of the problem. This also leads to more efficient methods of computing Lyapunov exponents since the linearisation of the system which is used to compute them can be decomposed on the isotypic components which are associated with the different irreducible representations of the group action.

We now extend the ideas of Aston and Dellnitz [6] to partial differential equations with symmetry. Again, the flow of a PDE is invariant on various fixed point spaces and,

Symmetry and Chaos in the Complex Ginzburg–Landau Equation. I: Reflectional Symmetries

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Abstract

The complex Ginzburg–Landau (CGL) equation on a 1–dimensional domain
w